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LETTER TO THE EDITOR

Planar model correlation functions†

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Abstract. Correlation functions are calculated on the critical line of the two-dimensional planar or XY model. A crossover is found between simple scaling behaviour for $2\pi K_{\text{eff}} > 4$, and more complex scaling behaviour for $2\pi K_{\text{eff}} = 4$. At the Kosterlitz–Thouless critical point, $2\pi K_{\text{eff}} = 4$, logarithmic factors appear in the correlation functions. These corrections to simple scaling behaviour are calculated for many correlation functions, including correlation functions not previously studied.

In previous discussions of the Kosterlitz-Thouless critical point, the behaviour of the order parameter correlation function has been examined in detail (Kosterlitz 1974, Amit *et al* 1980). The basic result is that for large r

$$\langle \cos \theta(\boldsymbol{r}) \cos \theta(\boldsymbol{0})\rangle \propto (\ln r)^{1/8}/r^{1/4}. \tag{1}$$

In this Letter we present an explanation of how such logarithmic factors appear in the limit $2\pi K_{\text{eff}} \to 4$, and use this insight to calculate the asymptotic forms of many other correlation functions.

The calculations are carried out in the generalised Villian model (Jose *et al* 1977, Kadanoff 1979). The parameters are K, the nearest-neighbour coupling constant, and y_0, the relative probability of a vortex excitation. When $y_0 = 0$ there are no vortices and the model reduces to the exactly soluble Gaussian spin wave model. When $y_0 > 0$ the model is in the planar or XY model universality class.

The main idea of the calculation is to use the Kosterlitz flow equations to resum the perturbation series for correlation functions. In terms of the variables x, proportional to $2\pi K - 4$, and y, proportional to y_0, the renormalisation group flow equations are (Kosterlitz 1974)

$$\mathrm{d}x/\mathrm{d}l = -\pi y^2, \qquad \mathrm{d}y/\mathrm{d}l = -4\pi xy \qquad (r \to r\mathrm{e}^{-l}). \tag{2}$$

The method for resumming perturbation series can be easily demonstrated for the correlation function of two $n = 2$ spin wave operators,

$$C(r, x, y) = \langle \cos 2\theta(r) \cos 2\theta(0)\rangle.$$

To first order in x and y, $C(r, x, y)$ has scaling dimension $1–2\pi x$. So $C(r, x, y)$ must obey the scaling equation

$$(\mathrm{d}/\mathrm{d}l)C(r, x, y) = (1 - 2\pi x)C(r, x, y). \tag{3}$$

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The structure of $C(r, x, y)$ obtained in perturbation theory is

$$C(r, x, y)=\frac{1}{r}\sum_{n,m=0}^{\infty}\sum_{x=0}^{n+m} a_{n,m,k}x^n y^m (\ln r)^{n+m-k}. \qquad (4)$$

By combining equations (2), (3) and (4) one obtains a recursion relation for the coefficients $a_{n,m,k}$. For $k=0$ we obtain

$$(n+m)a_{n,m,0}+\pi(n+1)a_{n+1,m-2,0}+(4\pi m-2\pi)a_{n-1,m,0}=0. \qquad (5)$$

With the initial conditions $a_{n,0,0}=(2\pi)^n/n!$, determined by the exact Gaussian result, and a_n, 1, $0=0$, determined by spin wave vortex 'charge' conservation, all the $a_{n,m,0}$ are determined. The unique (in this approximation) result is

$$C(r, x, y)=\frac{1}{r}\left(\cosh[2\pi(4x^2-y^2)^{1/2}\ln r]+\frac{2x}{(4x^2-y^2)^{1/2}}\sinh[2\pi(4x^2-y^2)^{1/2}\ln r]\right)^{1/2}. \qquad (6)$$

This agrees precisely with a third-order perturbation calculation.

Now we can examine the limit $2\pi K_{\text{eff}}\to 4$. When $2\pi K_{\text{eff}}>4$ $(4x^2-y^2>0)$ one obtains $C(r, x, y)=r^{-1+2\pi(x^2-y^2/4)^{1/2}}$ for large r; this is essentially the Gaussian result. But at the Kosterlitz–Thouless critical point, $2\pi K_{\text{eff}}=4$ $(y=2x)$, logarithms appear in the correlation function

$$C(r, x, y=2x)=(1+4\pi x\ \ln r)^{1/2}/r. \qquad (7)$$

These ideas can be extended to the case of n-point correlation functions. The result is $(\Sigma_i n_i=\Sigma_i m_i=0)$

$$\left\langle\prod_i O_{ni,mi}(r_i)\right\rangle =$$

$$\left\langle\prod_i O_{ni,mi}(r_i)\right\rangle_{x=y=0}\prod_{j<k}\left(\cosh[2\pi(4x^2-y^2)^{1/2}\ln r] +\frac{2x}{(4x^2-y^2)^{1/2}}\sinh[2\pi(4x^2-y^2)^{1/2}\ln r]\right)^{-2(n_jn_k-m_jm_k)}.$$

The first factor on the right-hand side is the exactly known Gaussian result at $2\pi K=4$. In the limit $y=2x$, equation (8) gives the leading logarithmic singularities in n-point correlation functions at the Kosterlitz–Thouless critical point.

Further details on an alternate derivation of these results and other universal corrections to correlation functions will be reported elsewhere.

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